



# Utilitarianism for infinite utility streams: A new welfare criterion and its axiomatic characterization<sup>☆</sup>

Kaushik Basu\*, Tapan Mitra

*Department of Economics, Cornell University, Ithaca, NY 14853, USA*

Received 7 February 2005; final version received 21 November 2005

Available online 25 January 2006

---

## Abstract

A definition of a utilitarian social welfare relation (SWR) for infinite utility streams is proposed. Such a relation is characterized in terms of the Pareto, Anonymity and Partial Unit Comparability Axioms. The merits of the utilitarian SWR, relative to the more restrictive SWR induced by the overtaking criterion, are examined.

© 2006 Elsevier Inc. All rights reserved.

*JEL classification:* D60; D70; D90

*Keywords:* Intergenerational equity; Pareto; Grading principle; Social welfare relations; Utilitarianism; Partial unit comparability; Overtaking criterion

---

## 1. Introduction

In comparing infinite utility streams, two guiding principles have generally been found to be widely acceptable. If, like Ramsey [28], we would like to treat all generations equally, we have to accept the Anonymity Axiom.<sup>1</sup> If our intertemporal preference structure is to be (positively)

---

<sup>☆</sup> The authors are grateful to Kuntal Banerjee, Wolfgang Buchholz and Jorgen Weibull for helpful suggestions on an earlier version of the paper. The current version has benefited greatly from the comments by a referee and an associate editor of the journal.

\* Corresponding author. Fax: +1 607 255 2818.

*E-mail addresses:* [kb40@cornell.edu](mailto:kb40@cornell.edu) (K. Basu), [tm19@cornell.edu](mailto:tm19@cornell.edu) (T. Mitra).

<sup>1</sup> Many authors have felt that a stronger notion than the Anonymity Axiom is needed to reflect intergenerational equity in intertemporal preferences. However, there appears to be general agreement that any notion of intergenerational equity in intertemporal preferences must include the Anonymity Axiom.

sensitive to the well-being of each generation, we are led to impose the Pareto Axiom.<sup>2</sup> It would be convenient if one could construct a *social welfare function* (SWF) which respected both of these principles, because then comparisons of infinite utility streams could be conveniently carried out in terms of the social welfare numbers associated with the utility streams.

In fact, it would be futile to try to construct such a SWF, because it can be shown that there is no SWF which respects both the Anonymity and the Pareto axioms. In other words, all Paretian SWFs are necessarily inequitable.<sup>3</sup>

This need not deter progress, however, because if one could construct a *social welfare ordering* (SWO) respecting the two axioms, we would be able to compare all infinite utility streams in terms of this ordering. Svensson [33] was the first to show that such an ordering does exist. However, it is worth noting that he obtains the ordering by non-constructive methods; specifically, he defines a pre-order (a binary relation satisfying reflexivity and transitivity) satisfying the two axioms, and then completes the order by appealing to Szpilrajn's lemma.<sup>4</sup> Thus, knowing that such an ordering exists does not necessarily provide a clue as to how it might be constructed.

In view of this, we might consider lowering our demands further and be willing to accept *social welfare relations* (SWRs) which are *pre-orders* that allow (consistent) comparisons between only some pairs of infinite utility streams but not others.<sup>5</sup> In this case, one can actually construct several SWRs satisfying the Anonymity and Pareto Axioms. The Suppes-Sen grading principle<sup>6</sup> and the pre-orders induced by the "overtaking" or "catching-up" criterion<sup>7</sup> are examples of such SWRs.

One way to be selective among such SWRs is to impose an axiom ensuring some degree of intertemporal comparability of utilities. In the context of intertemporal preferences, a partial (cardinal) unit comparability axiom appears to be a natural comparability requirement to have. If we do so, we obtain an interesting social welfare relation which compares only those infinite utility streams which are "Pareto comparable" beyond a finite horizon, and which applies standard utilitarian principles up to that finite horizon.

This *utilitarian SWR* satisfies the Anonymity, Pareto and Partial Unit Comparability axioms. It turns out that it is the *least restrictive* pre-order which does so.<sup>8</sup> If any SWR satisfies the

<sup>2</sup> The Rawlsian SWF, which figures quite prominently in discussions on equity, violates the Pareto principle, even in comparisons of utility streams where each utility stream has a well-defined minimum. For example, in comparing  $x$  and  $x'$ , where  $x_1 = x'_1 = 0.4$ , and  $x_n = 0.5 + (1/n)$  for  $n \geq 2$ ,  $x'_n = 0.5$  for  $n \geq 2$ ,  $x$  is clearly Pareto-superior to  $x'$ , but since  $\min_{n \geq 1} x_n = 0.4 = \min_{n \geq 1} x'_n$ , the Rawlsian SWF would consider the utility sequences to be indifferent.

<sup>3</sup> While this impossibility result might sound familiar, it has actually been established only recently in Basu and Mitra [6], without any domain restriction and without any other axiom imposed on preferences. The well-known impossibility result of Diamond [15] was established for a specific domain and, more importantly, with an additional continuity axiom on preferences.

<sup>4</sup> Recall that a standard way of proving Szpilrajn's Lemma is by using Zorn's Lemma, which is known to be equivalent to the Axiom of Choice. See, for example, Fishburn [16] for a proof of Szpilrajn's Lemma.

<sup>5</sup> Pre-orders, incomplete though they may be, have turned out to be powerful tools. The use of the Lorenz pre-order in studies of income inequality, and the pre-order induced by the overtaking criterion in optimal growth theory are two well-known examples.

<sup>6</sup> The Grading Principle is due to Suppes [32]. For a comprehensive analysis of it, see Sen [30].

<sup>7</sup> For the definitions of the SWRs appropriate to this discussion, see Section 4.

<sup>8</sup> To elaborate, if one identifies a binary relation with its graph, then the set  $S_U$  representing the graph of the utilitarian SWR is the smallest set (in terms of the partial order of  $\subset$ ) among all sets representing graphs of SWRs which satisfy the Anonymity, Pareto and Partial Unit Comparability Axioms. This is discussed fully in Section 3.1. The SWRs induced by the overtaking criterion (discussed in detail in Section 4) satisfy all three axioms, but they are clearly not the least restrictive SWRs which do so. The Suppes-Sen grading principle is, of course, a less restrictive pre-order than the utilitarian SWR, but it does not satisfy the Partial Unit Comparability axiom. This point is discussed in Section 3.2 below.

three axioms, then the utilitarian SWR is a *subrelation* to it in the sense that the rankings of the utilitarian SWR must always be respected by any such SWR. In this sense, the utilitarian SWR is *characterized* by the Anonymity, Pareto and Partial Unit Comparability axioms.<sup>9</sup>

We compare our utilitarian SWR with the SWRs induced by the overtaking or catching-up criteria in Section 4. A noteworthy feature of our utilitarian SWR is that it is axiomatized without postulating any continuity property on the pre-order in the infinite dimensional space containing the set of utility streams. In contrast, axiomatic characterizations of the more restrictive SWRs induced by the overtaking criterion typically involve some form of a continuity axiom.<sup>10</sup>

We argue that the rankings provided by our utilitarian SWR are more widely acceptable than the rankings provided by the overtaking SWR. Of course, the overtaking (and more so the catching-up) SWR provides rankings of two utility streams in many cases in which the utilitarian SWR finds them non-comparable. That is, the utilitarian SWR is more incomplete than the overtaking SWR. However, as an application of the utilitarian SWR, we establish, in the standard aggregative model of optimal growth without discounting, the somewhat surprising result that this incompleteness is not a handicap in characterizing dynamic optimal behavior, and the power of the overtaking (or catching-up) SWR to rank a larger set of utility streams than the utilitarian SWR is found to be completely superfluous.

## 2. Notation and definitions

Let  $\mathbb{N}$  denote, as usual, the set of natural numbers  $\{1, 2, 3, \dots\}$ , and let  $\mathbb{R}$  denote the set of real numbers. Let  $Y$  denote the closed interval  $[0, 1]$ , and let the set  $Y^{\mathbb{N}}$  be denoted by  $X$ . Then,  $X$  is the domain of utility sequences that we are interested in. Hence,  $x \equiv (x_1, x_2, \dots) \in X$  if and only if  $x_n \in [0, 1]$  for all  $n \in \mathbb{N}$ .

For  $y, z \in \mathbb{R}^{\mathbb{N}}$ , we write  $y \geq z$  if  $y_i \geq z_i$  for all  $i \in \mathbb{N}$ ; and, we write  $y > z$  if  $y \geq z$ , and  $y \neq z$ .

A SWR is a binary relation,  $\succsim$ , on  $X$ , which is reflexive and transitive (a pre-ordering).<sup>11</sup> We associate with  $\succsim$  its symmetric and asymmetric components in the usual way. Thus, we write  $x \sim y$  when  $x \succsim y$  and  $y \succsim x$  both hold; and, we write  $x \succ y$  when  $x \succsim y$  holds, but  $y \not\succsim x$  does not hold. A SWO is a binary relation,  $\succsim$ , on  $X$ , which is complete<sup>12</sup> and transitive (a complete pre-ordering).

A SWR  $\succsim_A$  is a *subrelation* to a SWR  $\succsim_B$  if (a)  $x, y \in X$  and  $x \succsim_A y$  implies  $x \succsim_B y$ ; and (b)  $x, y \in X$  and  $x \succ_A y$  implies  $x \succ_B y$ . A SWO  $\succsim_A$  is *compatible* with a SWR  $\succsim_B$  if and only if  $\succsim_B$  is a subrelation to  $\succsim_A$ .

Given  $x \in X$ , and  $N \in \mathbb{N}$ , let us denote by  $x(N)$  the vector consisting of the first  $N$  elements of  $x$  and by  $x[N]$  the sequence from term  $(N + 1)$  onwards. So,  $x(N) = (x_1, x_2, \dots, x_N)$  and  $x[N] = (x_{N+1}, x_{N+2}, \dots)$ . The sequence  $(x_1, x_2, \dots, x_N, 0, 0, \dots)$  is denoted by  $(x(N), 0[N])$ . Given a vector  $x(N)$ , we use  $I(x(N))$  to denote  $(x_1 + \dots + x_N)$ .

<sup>9</sup> In the same sense, the Suppes-Sen grading principle is characterized by the Anonymity and Pareto Axioms. See the discussion in Section 3.2.

<sup>10</sup> The study by Brock [9] uses a “consistency axiom” which, together with the independence axiom, actually implies a continuity restriction on the underlying preferences. A more recent study by Asheim and Tungodden [2] also uses a continuity axiom. This point is discussed in detail in Section 4.

<sup>11</sup> In the economics literature, a pre-ordering is often referred to as a “partial ordering” or as a “quasi ordering”. However, in the mathematics literature, the term “partial ordering” refers to a binary relation which is transitive and *antisymmetric*. To avoid confusion, we use the mathematical terminology, since the term “pre-order” is never used in any other sense in either discipline. Incidentally, our usage coincides with the terminology introduced in Debreu [13].

<sup>12</sup> Since completeness implies reflexivity, a SWO is a SWR, which is complete.

### 3. The utilitarian social welfare relation

In this section, we introduce a new definition of a *utilitarian* SWR, and provide an axiomatic characterization of it in terms of the Anonymity, Pareto and Partial Unit Comparability axioms. We also relate our utilitarian SWR to the Suppes-Sen grading principle, which is characterized in terms of the first two of these axioms.

Let us define a binary relation  $\succsim_U$  on  $X$  by:

$$x \succsim_U y \text{ if and only if there is } N \in \mathbb{N}, \text{ such that} \\ (I(x(N)), x[N]) \geq (I(y(N)), y[N]). \tag{1}$$

It is easy to check that  $\succsim_U$  is reflexive and transitive on  $X$ , so it is a SWR. We will call this SWR *utilitarian*. Note that the utilitarian SWR ranks only those infinite utility streams which are “Pareto comparable” beyond a finite horizon, and applies standard utilitarian principles up to that finite horizon.

The SWR  $\succsim_U$  satisfies the following two desirable properties:

$$(a) \text{ If } x, y \in X \text{ and } N \in \mathbb{N} \text{ and } (I(x(N)), x[N]) \geq (I(y(N)), y[N]) \text{ then } x \succsim_U y \tag{2}$$

and

$$(b) \text{ If } x, y \in X \text{ and } N \in \mathbb{N} \text{ and } (I(x(N)), x[N]) > (I(y(N)), y[N]) \text{ then } x \succ_U y. \tag{3}$$

The SWR  $\succsim_U$  also satisfies what has been called the “independent future” condition:

$$\text{If } x, y, z \in X \text{ then } x \succsim_U y \text{ if and only if } (z(N), x) \succsim_U (z(N), y) \text{ for every } N \in \mathbb{N}.$$

This condition follows from postulates 3b and 4 in Koopmans [19], and is explicitly stated in this form in Fleurbaey and Michel [17]. Thus, the passage of time does not alter the preferences, given a common history upto any point of time.

#### 3.1. Axiomatic characterization of the utilitarian SWR

Our objective is to establish an axiomatic characterization of the utilitarian SWR. To this end, consider the following two axioms on a SWR  $\succsim$ , which are fairly straightforward, and therefore require no explanation.

**Axiom 1 (Pareto).** If  $x, y \in X$ , and  $x > y$ , then  $x \succ y$ .

**Axiom 2 (Anonymity).** If  $x, y$  are in  $X$ , and there exist  $i, j$  in  $\mathbb{N}$ , such that  $x_i = y_j$  and  $x_j = y_i$ , while  $x_k = y_k$  for all  $k \in \mathbb{N}$ , such that  $k \neq i, j$ , then  $x \sim y$ .

The next axiom is an adaptation to the infinite domain of the standard assumption of unit interpersonal comparability used in social choice theory (see, for instance, Sen [31], d’Aspremont and Gevers [12], Roberts [29] and Basu [5]), expressed as an invariance axiom.<sup>13</sup>

<sup>13</sup> Maskin [22] uses the weaker “full comparability axiom” in which one demands invariance only for a *common* change of origin and a common change of scale for all agents. He is able to characterize utilitarianism (in finite societies) by using this “full comparability axiom” (instead of the stronger “unit comparability axiom” in d’Aspremont-Gevers [12]) by exploiting in addition a continuity axiom.

**Axiom 3** (Partial Unit Comparability). If  $x, y \in X, \alpha \in \mathbb{R}^{\mathbb{N}}$  and  $N \in \mathbb{N}$  satisfy:

$$x[N] = y[N] \quad \text{and} \quad x \succsim y \tag{4}$$

and

$$x + \alpha \in X, \quad y + \alpha \in X \tag{5}$$

then they must also satisfy:

$$x + \alpha \succsim y + \alpha. \tag{6}$$

**Remarks.** (i) The Unit Comparability axiom (on the infinite domain) asserts that preferences are invariant to changes in the origins of the utility indices used in the various periods; it is also invariant to a common change in the scale (by a positive factor) of the utility indices used in the various periods. It would be formally stated as follows.<sup>14</sup>

*Unit comparability:* Let  $a, b, a', b' \in X$  be such that there exists a sequence of real numbers  $\{\alpha_n\}$  and a positive real number  $\beta$  satisfying for all  $n \in \mathbb{N}$ ,

$$a'_n = \alpha_n + \beta a_n; \quad b'_n = \alpha_n + \beta b_n. \tag{7}$$

Then,

$$a \succ b \text{ if and only if } a' \succ b'. \tag{8}$$

(ii) Axiom 3 is weaker than the Unit Comparability axiom, since we insist on the invariance with respect to changes in origin only in comparing utility streams in which the streams are identical from a certain point onwards.<sup>15</sup>

It is fairly straightforward to check that if the utilitarian SWR  $\succsim_U$  is a subrelation to a SWR  $\succsim$ , then  $\succsim$  must satisfy the Pareto, Anonymity and Partial Unit Comparability Axioms. What is not so obvious is that if  $\succsim$  is any SWR satisfying these three axioms, then the utilitarian SWR  $\succsim_U$  must be a subrelation to  $\succsim$ . Essential to this complete characterization theorem is a technical lemma, which should be of independent interest. This intermediate result provides a characterization of the indifference classes (of SWRs satisfying the three axioms) on the subset of  $X$  consisting of utility streams with at most a finite number of non-zero entries.

Define:

$$X^0 = \{x \in X : x \text{ has at most a finite number of non-zero elements}\}. \tag{9}$$

Note that for  $x \in X^0$ , the sum  $\sum_{n=1}^{\infty} x_n$  is well-defined; we denote it by  $\sigma(x)$ . For  $x \in X^0$ , the decreasing rearrangement of  $x$  is clearly also well-defined; we denote it by  $\hat{x}$ . Define  $m(\hat{x}) = \min\{N \in \mathbb{N} : x_n = 0 \text{ for all } n \geq N\}$ .

**Lemma 1.** (i) Suppose a SWR  $\succsim$  satisfies Axioms 2, 3. If  $x, y \in X^0$ , and  $\sigma(x) = \sigma(y)$ , then  $x \sim y$ . (ii) Suppose a SWR  $\succsim$  satisfies Axioms 1–3. If  $x, y \in X^0$  and  $x \sim y$ , then  $\sigma(x) = \sigma(y)$ .

<sup>14</sup> This axiom has been used by Lauwers [21] in his axiomatic characterization of discounted utilitarianism.

<sup>15</sup> Weaker than the above partial unit comparability axiom is the *independence* postulate, introduced by Debreu [14] in the finite-horizon context, and studied by Koopmans [19] and Koopmans et al. [20] in an infinite-horizon context, in their studies on the representation of preferences by additively separable utility functions.

The proof of Lemma 1(i) (see Section 6) follows the method used by Milnor [24] in his axiomatic characterization of the Laplace criterion in games against nature. This idea has also been used in the context of social choice theory in characterizing utilitarianism in a society with a finite number of agents by d’Aspremont and Gevers [12]. For finite agent societies, a diagrammatic exposition of their result is provided in Blackorby et al. [7].

We now present our characterization result regarding the utilitarian SWR  $\succsim_U$ .

**Theorem 1.** *The utilitarian SWR  $\succsim_U$  is a subrelation to a SWR  $\succsim$  if and only if  $\succsim$  satisfies Axioms 1–3.*

It is useful to view the above characterization result as saying that the utilitarian SWR is the *least restrictive* SWR among all SWRs satisfying the Pareto, Anonymity and Partial Unit Comparability axioms. For this purpose, it is convenient to identify a binary relation on  $X$  with its graph in  $X^2$ .<sup>16</sup>

Denote by  $P(X^2)$  the set of all subsets of  $X^2$ . Note that the binary relation  $\subset$  (“subset of”) is a pre-order on  $P(X^2)$ . If  $\succsim$  is any binary relation on  $X$ , its graph:

$$S(\succsim) = \{(x, y) \in X^2 : x \succsim y\} \tag{10}$$

is a subset of  $X^2$ , and consequently,  $S(\succsim)$  is an element of  $P(X^2)$ .

We look at the subset  $Q(X^2)$  of  $P(X^2)$  consisting of the graphs of those pre-orders on  $X$  which satisfy Axioms 1–3. Formally,  $Q(X^2)$  is the subset of  $P(X^2)$ , defined by

$$Q(X^2) = \{S(\succsim) \in P(X^2) : \succsim \text{ is a pre-order on } X \text{ satisfying Axioms 1-3}\}. \tag{11}$$

Using Theorem 1, we have  $S(\succsim_U) \in Q(X^2)$ , and if  $\succsim$  is any pre-order on  $X$  satisfying Axioms 1–3, then  $S(\succsim_U) \subset S(\succsim)$ . Thus,  $S(\succsim_U)$  is a least element of  $Q(X^2)$  in terms of the pre-order  $\subset$ .<sup>17</sup> In this sense, the utilitarian SWR is the least restrictive pre-order satisfying Axioms 1–3.

It follows from this result that the graph of  $\succsim_U$  is in fact the intersection of the graphs of all pre-orders on  $X$  satisfying Axioms 1–3. Let us define

$$S = \bigcap_{S(\succsim) \in Q(X^2)} S(\succsim). \tag{12}$$

Then, by definition, we have  $S \subset S(\succsim)$  for every  $S(\succsim) \in Q(X^2)$ . Since  $S(\succsim_U) \in Q(X^2)$ , we have  $S \subset S(\succsim_U)$ . On the other hand, since  $S(\succsim_U)$  is a least element of  $Q(X^2)$  in terms of the pre-order  $\subset$ , we have  $S(\succsim_U) \subset S(\succsim)$  for all  $S(\succsim) \in Q(X^2)$ . That is,  $S(\succsim_U) \subset S$ . Thus, we have

$$S(\succsim_U) = S = \bigcap_{S(\succsim) \in Q(X^2)} S(\succsim). \tag{13}$$

### 3.2. Comparison with the grading principle

Recall that the Suppes-Sen *grading principle* is the binary relation  $\succsim_S$  defined<sup>18</sup> on  $X$  as follows:

$$x \succsim_S y \text{ if and only if there is a finite permutation } \pi \text{ of } \mathbb{N}, \text{ such that } x(\pi) \geq y.$$

<sup>16</sup> Recall that a binary relation is often *defined* precisely by specifying its graph.

<sup>17</sup> We use here the standard mathematical definition of a “least element” of a set, given a pre-order on that set. See, for example, Debreu [13, p.8].

<sup>18</sup> This formal definition of the Suppes-Sen grading principle in the context of infinite utility streams is due to Svensson [33].

It can be characterized as the least restrictive SWR satisfying the Pareto and Anonymity axioms; see d'Aspremont [11] and Asheim et al. [1].

**Proposition 1.** *A binary relation  $\succsim$  on  $X$  satisfies Axioms 1 and 2 if and only if the grading principle  $\succsim_S$  is a subrelation to  $\succsim$ .*

The grading principle does not satisfy the partial unit comparability axiom, as can be seen from the following example. Let  $x = (0.5, 0.4, 0.1, 0.1, \dots)$  and  $y = (0.3, 0.8, 0.1, 0.1, \dots)$ . Then,  $x$  and  $y$  are non-comparable by the Suppes-Sen grading principle, since there is no finite permutation of  $x$  which is  $\geq y$ , and there is no finite permutation of  $y$  which is  $\geq x$ . However, if we increase the utility origin of the first period by 0.1, and reduce the utility origin in the second period by 0.1, we obtain the vectors  $\bar{x} = (0.6, 0.3, 0.1, 0.1, \dots)$  and  $\bar{y} = (0.4, 0.7, 0.1, 0.1, \dots)$ . Now, permuting the first two periods of the vector  $\bar{y}$ , and denoting the resulting vector by  $\bar{y}(\pi)$ , we see that  $\bar{y}(\pi) > \bar{x}$ , so that  $\bar{y}$  is preferred to  $\bar{x}$  according to the Suppes-Sen grading principle. Thus, the Suppes-Sen grading principle violates the partial unit comparability axiom.

The characterizations of the grading principle and the utilitarian SWR allow us to obtain a SWO (a complete pre-order) compatible with the utilitarian SWR, which satisfies Anonymity and the Pareto axioms. Since the binary relation  $\succsim_U$  satisfies the Anonymity and Pareto axioms (by Theorem 1), the grading Principle  $\succsim_S$  is clearly a subrelation to  $\succsim_U$  (by Proposition 1). Thus, by Theorem 2 of Svensson [33, p.1253], there is a complete pre-order  $\succsim$  compatible with  $\succsim_U$ , which satisfies the Pareto and Anonymity Axioms.

#### 4. The overtaking criterion SWRs

The standard method of comparing utility streams in infinite-horizon intertemporal allocation models, while respecting the equal treatment of all generations, is by employing the overtaking criterion. The resulting pre-order is a generalization of the one used by Ramsey [28], and was proposed independently by Atsumi [3] and von Weizsacker [34] in their studies on optimal economic growth.

It would be useful to discuss the merits of our (less restrictive) utilitarian SWR with the (more restrictive) overtaking SWRs.<sup>19</sup> For this purpose, it would be convenient to have axiomatic characterizations of the overtaking SWRs which are directly comparable to our characterization of the utilitarian SWR. Unfortunately, the characterizations of the overtaking SWRs provided by Brock [9] and more recently by Asheim and Tungodden [2] use axiom sets which make such a direct comparison difficult.<sup>20</sup> In this section, we provide an axiomatic characterization of the overtaking SWRs, which will facilitate such a comparison.

We will show that the overtaking SWRs can be characterized in terms of Axioms 1–3 and an additional “consistency” axiom. This consistency axiom is similar in spirit to Axiom 3 used by Brock [9], who says that it “captures the notion that decisions on infinite programs are consistent with decisions on finite programs of length  $n$  if  $n$  is large enough.”

The partial unit comparability axiom together with the consistency axiom imply a continuity requirement on the SWR, similar to that used by Asheim and Tungodden [2] in their axiomatic

<sup>19</sup> A more appropriate way to describe them would be “the SWRs induced by the overtaking criterion and the catching-up criterion”. The precise definitions are given in Section 4.1.

<sup>20</sup> In particular, neither of these papers uses the partial unit comparability axiom directly.

characterization of the overtaking SWRs.<sup>21</sup> Thus, in terms of the axiomatics, the difference between the overtaking SWRs and our utilitarian SWR can be traced to imposing or not imposing a continuity requirement on preferences. In this regard, the current work can be seen as a continuation of the study in Basu and Mitra [6], where we deliberately refrained from imposing any continuity axiom. Axioms on the continuity of preferences in infinite-dimensional spaces have been the most controversial in the literature in this area, since the topology in which such continuity is assumed determines to a large extent the nature of allowable preferences. These restrictions arise largely from mathematical necessity and do not necessarily reflect any underlying ethical or economic principle.

4.1. Axiomatic characterization of the overtaking SWRs

There are two definitions of the overtaking criterion commonly in use. We will define each in turn and provide their axiomatic characterizations. Let us define<sup>22</sup> a binary relation  $\succsim_C$  on  $X$  by

$$x \succsim_C y \text{ if and only if there is } \bar{N} \in \mathbb{N}, \text{ such that} \\ I(x(N)) \geq I(y(N)) \text{ for all } N \geq \bar{N}. \tag{14}$$

It is easy to check that  $\succsim_C$  is reflexive and transitive on  $X$ , so it is a SWR. We will call it the *catching up SWR*.

The SWR  $\succsim_C$  satisfies the following two properties:

$$(a) \text{ If } x, y \in X \text{ and } \bar{N} \in \mathbb{N} \text{ and } I(x(N)) \geq I(y(N)) \\ \text{for all } N \geq \bar{N}, \text{ then } x \succsim_C y \tag{15}$$

and

$$(b) \text{ If } x, y \in X \text{ and } \bar{N} \in \mathbb{N} \text{ and } I(x(N)) \geq I(y(N)) \\ \text{for all } N \geq \bar{N}, \text{ and } I(x(N)) > I(y(N)) \\ \text{for a subsequence of } N \geq \bar{N}, \text{ then } x \succ_C y. \tag{16}$$

We can obtain an axiomatic characterization of the catching up SWR in terms of Axioms 1–3 of the previous section, and an additional *strong consistency* axiom, which we now state.

**Axiom 4 (Strong consistency).** For  $x, y \in X$ ,

$$(a) \text{ If there is } \bar{N} \in \mathbb{N}, \text{ such that } (x(N), 0[N]) \succsim (y(N), 0[N]) \\ \text{for all } N \geq \bar{N}, \text{ then } x \succsim y \tag{17a}$$

$$(b) \text{ If there is } \bar{N} \in \mathbb{N}, \text{ such that } (x(N), 0[N]) \succ (y(N), 0[N]) \\ \text{for all } N \geq \bar{N}, \text{ with } (x(N), 0[N]) \succ (y(N), 0[N]) \\ \text{for a subsequence of } N \geq \bar{N}, \text{ then } x \succ y \tag{17b}$$

The characterization result can be stated and proved as follows.

<sup>21</sup> This is discussed in detail in the discussion following Theorems 2 and 3.

<sup>22</sup> This definition is used by Svensson [33].



**Theorem 2.** A SWR  $\succsim$  satisfies Axioms 1–4 if and only if  $\succsim_C$  is a subrelation to  $\succsim$ .

**Remark.** Asheim and Tungodden [2] use a “Strong Preference Continuity” axiom in their characterization of the catching up SWR. This axiom is stated as follows.

*Strong preference continuity:* Suppose  $x, y \in X$  and  $\bar{N} \in \mathbb{N}$ , satisfy:

$$(y(N), x[N]) \succsim (x(N), x[N]) \quad \text{for all } N \geq \bar{N} \tag{Ca}$$

and:

$$(y(N), x[N]) \succ (x(N), x[N]) \quad \text{for a subsequence of } N \geq \bar{N} \tag{Cb}$$

then  $y \succ x$ .

This axiom is implied by Axioms 3 and 4. Using Axiom 3 and (Ca), (Cb), we have

$$(y(N), 0[N]) \succsim (x(N), 0[N]) \quad \text{for all } N \geq \bar{N} \tag{18}$$

and:

$$(y(N), 0[N]) \succ (x(N), 0[N]) \quad \text{for a subsequence of } N \geq \bar{N} \tag{19}$$

Thus, by Axiom 4, we obtain  $y \succ x$ .

To see that this is indeed a strong continuity requirement, define the sequence of infinite utility streams  $z^s = (y(s), x[s])$  for each  $s \in \mathbb{N}$ . Then, we see that  $y$  is the (pointwise) limit of  $z^s$  as  $s \rightarrow \infty$ . Thus, the axiom says that if  $z^s \succsim x$  for all  $s$  sufficiently large, with  $z^s \succ x$  for a subsequence of  $s$ , then the pointwise limit of  $z^s$  (namely,  $y$ )  $\succ x$ . Quite apart from the fact that preferences cannot be reversed in the limit, this in fact demands that strict preference prevail in the limit.<sup>23</sup>

The alternative version of the overtaking criterion<sup>24</sup> can be formalized as follows. Let us define the binary relation  $\succsim_O$  by

$$\begin{aligned} x \succsim_O y \text{ if and only if} \\ \text{either (i) } \exists \bar{N} \in \mathbb{N}, \text{ such that } I(x(N)) > I(y(N)) \quad \text{for all } N \geq \bar{N} \\ \text{or (ii) } \exists \bar{N} \in \mathbb{N}, \text{ such that } I(x(N)) = I(y(N)) \quad \text{for all } N \geq \bar{N}. \end{aligned} \tag{20}$$

It is easy to check that  $\succsim_O$  is reflexive and transitive on  $X$ , so it is a SWR. We will call it the *overtaking SWR*.

It can be verified that the SWR  $\succsim_O$  satisfies the following two properties:

$$\begin{aligned} \text{(a) } x \succ_O y \text{ if and only if there is } \bar{N} \in \mathbb{N}, \\ \text{such that } I(x(N)) > I(y(N)) \text{ for all } N \geq \bar{N} \end{aligned} \tag{21}$$

and

$$\begin{aligned} \text{(b) } x \sim_O y \text{ if and only if there is } \bar{N} \in \mathbb{N}, \\ \text{such that } I(x(N)) = I(y(N)) \text{ for all } N \geq \bar{N}. \end{aligned} \tag{22}$$

<sup>23</sup> Asheim and Tungodden [2] use the word “strong” here because strict preference in the limit is based on strict preference holding only along a subsequence of  $s$ . But, clearly, this continuity requirement is very strong for other reasons as well.

<sup>24</sup> This is the version used by Atsumi [3], von Weizsacker [34] and Brock [9].

It follows that

$$(i) x \succ_O y \implies x \succ_C y; \quad (ii) x \sim_O y \implies x \sim_C y. \tag{23}$$

We can obtain an axiomatic characterization of the strong overtaking SWR in terms of Axioms 1–3 of the previous section, and an additional *weak consistency* axiom, which we now state.

**Axiom 5 (Weak consistency).** For  $x, y \in X$ ,

$$(a) \text{ If there is } \bar{N} \in \mathbb{N}, \text{ such that } (x(N), 0[N]) \sim (y(N), 0[N]) \\ \text{for all } N \geq \bar{N}, \text{ then } x \sim y \tag{24a}$$

$$(b) \text{ If there is } \bar{N} \in \mathbb{N}, \text{ such that } (x(N), 0[N]) \succ (y(N), 0[N]) \\ \text{for all } N \geq \bar{N}, \text{ then } x \succ y. \tag{24b}$$

The characterization result for the overtaking SWR can be stated as follows. The proof, which is similar to the proof of Theorem 2, is omitted.

**Theorem 3.** A SWR  $\succsim$  satisfies Axioms 1–3 and 5 if and only if  $\succsim_O$  is a subrelation to  $\succsim$ .

**Remark.** Asheim and Tungodden [2] use a “Weak Preference Continuity” axiom in their characterization of the overtaking SWR. This axiom is stated as follows.

*Weak preference continuity:* Suppose  $x, y \in X$  and  $\bar{N} \in \mathbb{N}$ , satisfy:

$$(y(N), x[N]) \succ (x(N), x[N]) \quad \text{for all } N \geq \bar{N} \tag{Cw}$$

then  $y \succ x$ .

This axiom is implied by Axioms 3 and 5. Using Axiom 3 and (Cw), we have

$$(y(N), 0[N]) \succsim (x(N), 0[N]) \quad \text{for all } N \geq \bar{N}.$$

Further

$$(x(N), 0[N]) \succsim (y(N), 0[N])$$

cannot hold for any  $N \geq \bar{N}$ . For if it did hold for some  $N \geq \bar{N}$ , then by Axiom 3, we would get

$$(x(N), x[N]) \succsim (y(N), x[N])$$

for that  $N$ . But, this would contradict (Cw). Thus, we have

$$(y(N), 0[N]) \succ (x(N), 0[N]) \quad \text{for all } N \geq \bar{N}$$

and, by Axiom 5, we obtain  $y \succ x$ .

Define the sequence of infinite utility streams  $z^s = (y(s), x[s])$  for each  $s \in \mathbb{N}$ . Then, we see that  $y$  is the (pointwise) limit of  $z^s$  as  $s \rightarrow \infty$ . Thus, the axiom says that if  $z^s \succ x$  for all  $s$  sufficiently large, then the pointwise limit of  $z^s$  (namely,  $y$ )  $\succ x$ .

A corollary of the characterization results on the utilitarian and the overtaking SWRs (Theorems 1–3), which is useful for our discussion in Sections 4.2 and 5, can be stated as follows.

**Corollary 1.** *The utilitarian SWR  $\succsim_U$  is a subrelation to the overtaking SWR  $\succsim_O$ , which in turn is a subrelation to the catching up SWR  $\succsim_C$ .*

4.2. *The utilitarian SWR versus the overtaking SWRs*

We find the ranking of utility streams according to the utilitarian SWR to be persuasive. Consider a situation in which faced with a choice between  $x$  and  $y$ , one finds that there is some  $N' \in \mathbb{N}$ , such that

$$(I(x(N')), x[N']) > (I(y(N')), y[N']).$$

Then one can find  $N \geq N'$  such that

$$I(x(N)) > I(y(N)) \quad \text{and} \quad x[N] \geq y[N].$$

Thus, one may consider getting together the members of the finite society  $\{1, \dots, N\}$ , and asking them to rank  $x$  versus  $y$ . If they apply utilitarian principles to themselves, they will rank  $x$  above  $y$ . In this case, it is legitimate for the *infinite-horizon society* to rank  $x$  above  $y$  because the infinite number of future generations, who are *not* included in the finite society  $\{1, \dots, N\}$ , are either indifferent between  $x$  and  $y$  or prefer  $x$  to  $y$ . In other words, in this situation, all future generations beyond  $N$  are willing to go along with the (utilitarian) preferences of the finite society  $\{1, \dots, N\}$ .

No such consensus is to be obtained with the overtaking SWR. Consider the following example of two utility streams, where the overtaking SWR can compare the two streams and the utilitarian SWR declares them non-comparable.

$$\left. \begin{aligned} x &= (0.2, 0, \quad 0.1, 0, \quad 0.1, 0, \quad \dots), \\ y &= (0, \quad 0.1, 0, \quad 0.1, 0, \quad 0.1, \dots). \end{aligned} \right\}$$

We can verify that for  $\bar{N} = 1$ ,

$$I(x(N)) > I(y(N)) \quad \text{for all } N \geq \bar{N}$$

so that  $x \succ y$  according to the overtaking SWR.

The question arises whether  $x$  *should* be preferred to  $y$  by the infinite horizon society. This is not altogether clear. The problem with judging  $x \succ y$  in such a case can be seen as follows. If we look at any finite-horizon society, and ask the society to rank  $x$  versus  $y$ , they will indeed rank  $x$  higher than  $y$ , if they apply utilitarian principles to themselves. However, no matter how large the finite-horizon, there are always an *infinite* number of future generations who rank  $x$  below  $y$ . Thus, it is never possible to have consensus of opinion between any finite-horizon society and the infinite number of future generations not included in that finite society.<sup>25</sup>

A comparison of the utilitarian SWR with the overtaking or catching-up SWR can also be made from a somewhat different perspective. Recall that the need to construct such SWRs arises because SWFs satisfying Pareto and Anonymity axioms do not exist (and no SWO satisfying the two axioms has been constructed). If one discounted future utilities one could get a SWF

---

<sup>25</sup> It is possible to use an extended notion of Anonymity, which does not conflict with the Pareto principle, which makes comparisons of utility streams, of the sort described in the example, possible. Such an extended notion of Anonymity leads to an extended grading principle and an extended utilitarian SWR, which are more complete SWRs than the grading principle and the utilitarian SWR (respectively) discussed in this paper. For results along these lines, see the recent papers by Mitra and Basu [26] and Banerjee [4].

(in our setting) by simply summing up the discounted stream of utilities. The Pareto Axiom would be satisfied but the Anonymity axiom would, of course, be violated. Loosely speaking, this violation could be considered to be “small” for discount factors close to 1. Thus, one way of checking for robustness of a SWR, satisfying Anonymity and the Pareto axioms, would be to see whether the ranking between two alternatives  $x$  and  $y$  provided by the SWR is preserved for discount factors close to 1 in the discounted present value SWF.<sup>26</sup>

Note that if  $x$  is preferred to  $y$  according to the utilitarian SWR, then there is some  $N \in \mathbb{N}$ , such that  $I(x(N)) > I(y(N))$  and  $x[N] \geq y[N]$ . Thus, there is a discount factor  $\hat{\delta} \in (0, 1)$ , such that for all  $\delta \in (\hat{\delta}, 1)$ , we would have

$$\sum_{n=1}^N \delta^{n-1} x(n) > \sum_{n=1}^N \delta^{n-1} y(n) \quad \text{and} \quad \delta^{n-1} x(n) \geq \delta^{n-1} y(n) \quad \text{for } n \geq N + 1.$$

Consequently, for all  $\delta \in (\hat{\delta}, 1)$ , we would have  $f(x; \delta) > f(y; \delta)$ , where  $f(\cdot, \delta)$  is the discounted present value SWF, corresponding to the discount factor  $\delta$ .

The overtaking SWR does *not* have this robustness property, and we show this by presenting a concrete example of two utility streams  $x$  and  $y$ , such that  $x$  is preferred to  $y$  according to the overtaking SWR, but  $y$  is preferred to  $x$  according to the discounted present value SWF for every  $\delta \in (0, 1)$ .<sup>27</sup>

Define  $x$  and  $y$  as follows:

$$\begin{aligned} x &= (0, \quad 0.5 + a, \quad a^2, \quad a^3, \quad a^4, \quad a^5, \quad \dots), \\ y &= (0.5, \quad 0, \quad a, \quad a^2, \quad a^3, \quad a^4, \quad \dots), \end{aligned}$$

where  $a = (1/8)$ . Denoting  $I(y(N)) - I(x(N))$  by  $A_N$ , we see that

$$A_1 = 0.5, \quad A_N = -a^{N-1} \quad \text{for all } N \geq 2. \tag{25}$$

Clearly then we have

$$I(x(N)) > I(y(N)) \quad \text{for all } N \geq 2$$

and consequently  $x \succ_O y$ .

We now claim that for the above example, for all  $\delta \in (0, 1)$ ,

$$f(x, \delta) \equiv \sum_{n=1}^{\infty} \delta^{n-1} x(n) < \sum_{n=1}^{\infty} \delta^{n-1} y(n) \equiv f(y, \delta). \tag{26}$$

Suppose, on the contrary, there is some  $\delta \in (0, 1)$ , such that

$$f(x, \delta) \geq f(y, \delta). \tag{27}$$

Given this  $\delta \in (0, 1)$ , denote  $(1 - \delta)/2$  by  $\beta$ ; then  $\beta > 0$ . We can choose  $\bar{N} \in \mathbb{N}$  large enough so that

$$\delta^{\bar{N}} / (1 - \delta) \leq (\beta/2). \tag{28}$$

<sup>26</sup> This robustness check was suggested to us by Jorgen Weibull.

<sup>27</sup> Although the overtaking criterion has been discussed at length in the literature, we are not aware of any paper which presents such an example.

Using (28), note that for all  $N \geq \bar{N}$ , we have

$$\sum_{n=N+1}^{\infty} \delta^{n-1}x(n) \leq \sum_{n=N+1}^{\infty} \delta^{n-1} = \delta^N/(1-\delta) \leq \delta^{\bar{N}}/(1-\delta) \leq (\beta/2). \tag{29}$$

Using (27) and (29), note that for all  $N \geq \bar{N}$ , we have

$$\begin{aligned} \sum_{n=1}^N \delta^{n-1}x(n) &= \sum_{n=1}^{\infty} \delta^{n-1}x(n) - \sum_{n=N+1}^{\infty} \delta^{n-1}x(n) \\ &= f(x, \delta) - \sum_{n=N+1}^{\infty} \delta^{n-1}x(n) \\ &\geq f(y, \delta) - (\beta/2) \\ &\geq \sum_{n=1}^N \delta^{n-1}y(n) - (\beta/2). \end{aligned}$$

Thus, for all  $N \geq \bar{N}$ , we obtain

$$\sum_{n=1}^N \delta^{n-1}y(n) - \sum_{n=1}^N \delta^{n-1}x(n) \leq (\beta/2). \tag{30}$$

Using (25), we now write for all  $N \in \mathbb{N}$ ,

$$\begin{aligned} \sum_{n=1}^N \delta^{n-1}(y(n) - x(n)) &= A_1 + (A_2 - A_1)\delta + \dots + (A_N - A_{N-1})\delta^{N-1} \\ &= A_1(1-\delta) + \dots + A_{N-1}\delta^{N-2}(1-\delta) + A_N\delta^{N-1} \\ &= 0.5(1-\delta) - (1-\delta)[\delta a + \delta^2 a^2 \\ &\quad + \dots + \delta^{N-2}a^{N-2}] - \delta^{N-1}a^{N-1} \\ &\geq 0.5(1-\delta) - (1-\delta) \sum_{n=1}^{\infty} \delta^n a^n - \delta^{N-1}a^{N-1} \\ &= 0.5(1-\delta) - (1-\delta)[\delta a/(1-\delta a)] \\ &\quad - \delta^{N-1}a^{N-1}. \end{aligned} \tag{31}$$

We have

$$[\delta a/(1-\delta a)] \leq [a/(1-a)] = (1/7)$$

and using this information in (31), we obtain for all  $N \in \mathbb{N}$ ,

$$\sum_{n=1}^N \delta^{n-1}(y(n) - x(n)) \geq (5/14)(1-\delta) - \delta^{N-1}a^{N-1}. \tag{32}$$

Combining (30) and (32), we obtain for all  $N \geq \bar{N}$ ,

$$(1-\delta)/4 = (\beta/2) \geq (5/14)(1-\delta) - \delta^{N-1}a^{N-1}.$$

This means that for all  $N \geq \bar{N}$ , we have

$$\delta^{N-1} a^{N-1} \geq (3/28)(1 - \delta). \quad (33)$$

Note that the right-hand side of (33) is a positive constant (since  $\delta \in (0, 1)$  is given) independent of  $N$ . The left-hand side of (33) depends on  $N$ , and goes to zero as  $N \rightarrow \infty$ . This contradiction establishes our claim (26). That is  $y$  is preferred to  $x$  according to the present discounted value SWF for every  $\delta \in (0, 1)$ .

In fact, for this example, the sums  $f(x, \delta)$  and  $f(y, \delta)$  exist even for  $\delta = 1$ , and it is easy to check that

$$f(x, 1) \equiv \sum_{n=1}^{\infty} x(n) = \sum_{n=1}^{\infty} y(n) \equiv f(y, 1)$$

so that, according to utilitarian principles, certainly one should declare  $x$  to be indifferent to  $y$ , but as we have noted above,  $x$  is strictly preferred to  $y$  according to the overtaking criterion.<sup>28</sup>

## 5. An application to optimal growth theory

We have argued above that Axioms 4 or 5 are not as readily acceptable as Axioms 1–3; in other words, we find the overtaking and catching-up SWRs less persuasive than the utilitarian SWR. Nevertheless, almost all of the theory of optimal intertemporal allocation, in which generations are treated equally in its preference structure, uses some form of the overtaking or catching up SWR, and therefore accepts Axiom 4 or 5 (in addition to Axioms 1–3). The reason for this is that even though Axiom 4 (or 5) is not an obvious axiom to accept, it gives sufficient structure to intertemporal preferences so that the theory of optimal intertemporal allocation has some predictive power: a path which is optimal according to this pre-ordering in the typical intertemporal model is unique, and the nature of such an optimal path can be described quite accurately, both in terms of short-run characteristics (the *Ramsey–Euler* or *competitive* conditions), and long-run behavior (the *turnpike* property).

The presumption appears to be that if we wanted to proceed with intertemporal preferences satisfying only Axioms 1–3 (that is, *without* imposing something like Axiom 4 or 5), we would not have a useful theory of optimal behavior over time. Since the utilitarian SWR is less complete, it would allow less comparisons of intertemporal paths, and consequently there could be many maximal points according to the utilitarian SWR. However, this issue has not been explored in the literature, and therefore such misgivings about a theory based solely on the utilitarian SWR might be premature.

We will establish in this section the rather surprising result that in the standard neoclassical model of optimal growth without discounting, *a maximal path according to the utilitarian SWR is in fact unique and it overtakes all other paths starting from the same initial condition*. This shows that the additional power of comparison, gained by using the overtaking SWR or catching up SWR, is redundant in this context.<sup>29</sup>

<sup>28</sup> We owe this remark to Wolfgang Buchholz.

<sup>29</sup> It is of interest to enquire whether this result generalizes to optimal growth models with heterogenous capital goods. While this general question has not been settled, Mitra [25] has shown that, in a model of forestry (which is a specific instance of a model with heterogenous capital goods), the result holds.

We begin by describing the standard one-good model of optimal growth, where future utilities are not discounted relative to the present. The framework is described by a pair of functions  $(f, u)$ , where  $f$  is the production function and  $u$  the utility function.

The production function,  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  will be supposed to satisfy the following assumptions:

- (F.1)  $f(0) = 0$ ,  $f$  is increasing, continuous and concave on  $\mathbb{R}_+$ .
- (F.2)  $f$  is twice continuously differentiable on  $\mathbb{R}_{++}$ , with  $f'(k) > 0$  and  $f''(k) < 0$  for  $k > 0$ .
- (F.3)  $\lim_{k \rightarrow 0} [f(k)/k] > 1$  and  $\lim_{k \rightarrow \infty} [f(k)/k] < 1$ .

The utility function,  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  will be supposed to satisfy the following assumptions:

- (U.1)  $u$  is increasing, continuous and concave on  $\mathbb{R}_+$ .
- (U.2)  $u$  is twice continuously differentiable on  $\mathbb{R}_{++}$ , with  $u'(c) > 0$  and  $u''(c) < 0$  for  $c > 0$ .
- (U.3)  $u'(c) \rightarrow \infty$  as  $c \rightarrow 0$ .

It can be shown that there exist uniquely determined numbers  $\bar{k}$  and  $\hat{k}$  such that  $0 < \hat{k} < \bar{k} < \infty$ , and  $f(\bar{k}) = \bar{k}$ ,  $f'(\hat{k}) = 1$ . We refer to  $\bar{k}$  as the *maximum sustainable stock* and to  $\hat{k}$  as the *golden-rule stock*.

A *feasible path* from  $k \geq 0$  is a sequence of *capital stocks*  $\{k_t\}$  satisfying:

$$k_0 = k, \quad 0 \leq k_{t+1} \leq f(k_t) \text{ for } t \geq 0.$$

Associated with the feasible path  $\{k_t\}$  from  $k$  is a *consumption* sequence  $\{c_t\}$ , defined by

$$c_t = f(k_{t-1}) - k_t \text{ for } t \geq 1$$

and a *utility* sequence  $\{x_t\}$ , defined by

$$x_t = u(c_t) \text{ for } t \geq 1.$$

It is easy to show that for every feasible path  $\{k_t\}$  from  $k \geq 0$ , we have

$$k_t \leq M(k) \text{ for } t \geq 0; \quad c_t \leq M(k) \text{ for } t \geq 1, \tag{34}$$

where  $\max\{\bar{k}, k\} \equiv M(k)$ . We will confine our discussion to feasible paths starting from initial stocks  $k \in [0, \bar{k}]$ . Then, since  $M(k) = \bar{k}$ , it follows that for every feasible path  $\{k_t\}$  from  $k \in [0, \bar{k}]$ , we have

$$k_t \leq \bar{k} \text{ for } t \geq 0; \quad c_t \leq \bar{c} \text{ for } t \geq 1; \quad x_t \leq u(\bar{c}) \text{ for } t \geq 1.$$

Thus, utility sequences associated with feasible paths from  $k \in [0, \bar{k}]$  belong to  $X = [0, 1]^{\mathbb{N}}$ , as in our framework of Section 2, if we normalize  $u(0) = 0$  and  $u(\bar{c}) = 1$ . (Note that such a normalization does not change the ranking of feasible paths according to the utilitarian, catching-up or overtaking SWRs.)

We will say that a feasible path  $\{k_t\}$  from  $k_o$  is *maximal* if there is no feasible path  $\{k'_t\}$  from  $k_o$  such that

$$x' \succ_U x$$

That is, in terms of the pre-order  $\succsim_U$ , the utility sequence  $\{x_t\}$  associated with the feasible path  $\{k_t\}$  from  $k_o$  is a *maximal element* among all sequences  $\{x'_t\}$  associated with feasible paths  $\{k'_t\}$  from  $k_o$ .<sup>30</sup>

<sup>30</sup> Using the result of Brock [8], one can infer that there exists a maximal path in our framework.

Note that  $\succsim_U$  is a pre-order which allows us to compare fewer utility sequences than the catching up pre-order  $\succsim_C$  or the overtaking pre-order  $\succsim_O$ . Nevertheless, we will show that a maximal path actually has a very strong property: it overtakes every other feasible path starting from the same initial stock. That is, if the feasible path  $\{k_t\}$  from  $k_o$  is maximal, then for every other feasible path  $\{k'_t\}$  from  $k_o$ , we have

$$x \succ_O x'$$

so that  $\{x_t\}$  is in fact the *greatest element*, in terms of the strict overtaking pre-order  $\succ_O$ , among all sequences  $\{x'_t\}$  associated with feasible paths  $\{k'_t\}$  from  $k_o$ . Thus, the pre-order  $\succsim_U$  is sufficient for a completely satisfactory theory of optimal growth, making the additional (often dubious) comparisons, made possible by the catching-up or overtaking pre-orders, quite superfluous.

**Theorem 4.** *Let  $\{\bar{k}_t\}$  be any maximal path from  $k_o \in (0, \bar{k}]$ , in the aggregative neoclassical model, with associated utility sequence  $\{\bar{x}_t\}$ . Then,*

$$\bar{x} \succ_O x',$$

where  $\{x'_t\}$  is the utility sequence associated with any other feasible path  $\{k'_t\}$  from  $k_o$ .

The proof of this result (presented in the next section) is fairly long and involved, so we provide here the basic logic of it.

A maximal element of our utilitarian SWR has two features. First, given an arbitrary *finite horizon*, if a maximal path is compared with any other path, which is identical to the maximal path beyond this finite horizon, then the sum of utilities over the finite horizon must be maximized at the maximal path. This yields the information (in the neoclassical growth model) that the maximal path must satisfy the *Ramsey–Euler conditions* for all time, being simply the first-order conditions of the appropriate maximization problem. Second, the maximal path cannot be Pareto dominated by any other path, so that the maximal path must be *efficient* in terms of the consumption sequence (generating the utility sequence), since the utility function is increasing. These two features imply, by a beautiful characterization result of Brock [10], that there is no path which can overtake a maximal path by a positive finite amount.<sup>31</sup> This is Step 1 in the proof, and it is where one can see clearly how the two features of the utilitarian SWR are used. (The first step applies to all growth models for which Brock’s characterization result is valid, and this is a larger class of models than is considered here, including in particular, models with changing technology over time.)

To infer from this result that a maximal path in fact overtakes every other path (starting from the same initial conditions) takes a considerable amount of additional work; specifically, it involves seeing the full implications of the result (obtained in Step 1) for the particular growth model examined here. The line of argument followed here crucially uses the stationary structure of the growth model, and the strict concavity of the production and utility functions, and can be conveniently subdivided into two parts.

The first part (Steps 2–6 in the proof) derives *asymptotic* properties of paths which are not “infinitely worse” (in terms of the sum of utilities) than a particular stationary path called a *golden-rule path*. These paths are called *good*, and their long-run properties can be derived solely by using the (stationary) shadow-prices associated with the golden-rule path, a method principally

<sup>31</sup> Paths which have this feature are called “weakly maximal” by Brock [8,10]. We deliberately refrain from using this terminology in this paper to avoid confusion.



due to Brock [8]. The idea then is to note that a maximal path is necessarily good, and therefore has precisely all the “right” long-run properties, that a path which was optimal according to the usual overtaking (or catching-up) criteria would have.

This still leaves open the possibility that a maximal path might misallocate resources over time in *transition* compared to a path which was optimal according to the usual overtaking (or catching-up) criteria. The second part (Step 7) of the argument can be seen to demonstrate that this possibility can be ruled out, by following a method that is essentially due to Gale [18]. This is accomplished by using the Ramsey–Euler equations to provide a (typically non-stationary) price support for the maximal path itself, and then exploiting the long-run properties of *all good paths*, to conclude that any deviation from the maximal path leads to a loss, in terms of the sum of utilities over a certain finite horizon, which can never be recovered in the future beyond that horizon.

**6. Proofs**

**Proof of Lemma 1.** (i) Let  $m = \max\{m(\hat{x}), m(\hat{y})\}$ . Then, for  $n \geq m$ , we have  $\hat{x}_n = \hat{y}_n = 0$ . We prove the result by induction on  $m$ . For  $m = 1$ , we have  $x = y = 0$ , and the result is trivially true. Suppose, next, that the result is true for  $m = 1, \dots, M$ , where  $M \in \mathbb{N}$ . We want to prove that the result is true for  $m = M + 1$ .

Suppose  $x, y \in X$  such that  $\max\{m(\hat{x}), m(\hat{y})\} = M + 1$ . If  $\hat{x} = \hat{y}$ , then  $x \sim y$  by the Anonymity Axiom. So, we need only consider the case in which  $\hat{x} \neq \hat{y}$ . Then, there exist some  $i, j \in \{1, \dots, M\}$ , such that  $\hat{x}_i > \hat{y}_i$  and  $\hat{x}_j < \hat{y}_j$ . Define  $\alpha_n = \min\{\hat{x}_n, \hat{y}_n\}$ ,  $x'_n = \hat{x}_n - \alpha_n$  and  $y'_n = \hat{y}_n - \alpha_n$  for all  $n \in \mathbb{N}$ . Notice that  $x', y' \in X^0$ , and  $x' \leq \hat{x}$ , while  $y' \leq \hat{y}$ . Further, we have  $\sigma(x') = \sigma(y')$ . Also,  $y'_i = 0$ , and  $x'_j = 0$ ; furthermore,  $x'_n = y'_n = 0$  for all  $n \geq M + 1$ . It follows that  $m(\hat{x}') \leq M$  and  $m(\hat{y}') \leq M$ ; consequently  $\max\{m(\hat{x}'), m(\hat{y}')\} \leq M$ . Thus by the induction hypothesis,  $\hat{x}' \sim \hat{y}'$ , and by the Anonymity axiom, we must have  $x' \sim y'$ . Now, using the Partial Unit Comparability axiom, we get  $\hat{x} \sim \hat{y}$ . The Anonymity Axiom can now be employed again to conclude that  $x \sim y$ . This completes the proof of (i) by induction.

(ii) Suppose, on the contrary, there exist  $x, y \in X^0$  satisfying  $x \sim y$ , but  $\sigma(x) \neq \sigma(y)$ . Without loss of generality, we may suppose that  $\sigma(x) > \sigma(y)$ ; denote  $[\sigma(x) - \sigma(y)]$  by  $d$ . Clearly, there exists  $N \in \mathbb{N}$ , such that  $x_i = y_i = 0$  for all  $i > N$ . Then, we have  $0 < d \leq N$ . Define  $x' \in \mathbb{R}^{\mathbb{N}}$  as follows:

$$\left. \begin{aligned} x'_i &= y_i && \text{for } i = 1, \dots, N, \\ x'_i &= (d/N) && \text{for } i = N + 1, \dots, 2N, \\ x'_i &= 0 && \text{for } i > 2N. \end{aligned} \right\}$$

Clearly,  $x' \in X$  and  $x' > y$ , so by the Pareto axiom,  $x' \succ y$ . It is also clear that  $x' \in X^0$ , and  $\sigma(x') = \sigma(y) + d = \sigma(x)$ . Thus, by part (i) of the Lemma, we have  $x' \sim x$ . Since  $x \sim y$ , we must have  $x' \sim y$ , a contradiction, which establishes (ii).  $\square$

**Proof of Theorem 1.** *Necessity:* Suppose the utilitarian SWR  $\succsim_U$  is a subrelation to a SWR  $\succsim$ . We need to verify that  $\succsim$  satisfies Axioms 1–3. To verify that  $\succsim$  satisfies the Pareto axiom, let  $x, y \in X$ , such that there is some  $j \in \mathbb{N}$  for which  $x_j > y_j$ , while  $x_k \geq y_k$  for all  $k \neq j$ . Then, clearly, we have  $(I(x(j)), x[j]) > (I(y(j)), y[j])$ , so  $x \succ_U y$  by (3). Since  $\succsim_U$  is a subrelation to  $\succsim$ , we have  $x \succ y$ . To verify that  $\succsim$  satisfies the Anonymity axiom, let  $x, y \in X$ , and  $i, j \in \mathbb{N}$  be such that  $x_i = y_j$  and  $x_j = y_i$ , while  $x_k = y_k$  for all  $k \in \mathbb{N}$ , such that  $k \neq i, j$ . Then, defining  $N = \max\{i, j\}$ , we have  $(I(x(N)), x[N]) = (I(y(N)), y[N])$ , so that  $x \succsim_U y$  and  $y \succsim_U x$  by (2).

Since  $\succsim_U$  is a subrelation to  $\succsim$ , we have  $x \succsim y$  and  $y \succsim x$ . Thus,  $x \sim y$ , as required. To verify the Partial Unit Comparability axiom, let  $x, y \in X, \alpha \in \mathbb{R}^{\mathbb{N}}$  and  $N \in \mathbb{N}$  satisfy

$$(x(N), x[N]) \succsim (y(N), x[N]) \tag{35}$$

and

$$(x(N), x[N]) + \alpha \in X, \quad (y(N), x[N]) + \alpha \in X. \tag{36}$$

We claim that  $I(x(N)) \geq I(y(N))$ . For if  $I(x(N)) < I(y(N))$ , then by (3), we have  $(y(N), x[N]) \succ_U (x(N), x[N])$ . Since  $\succsim_U$  is a subrelation to  $\succsim$ , we must then have  $(y(N), x[N]) \succ (x(N), x[N])$ , a contradiction to (35), which establishes our claim. Thus, we have

$$I(x(N) + \alpha(N)) = I(x(N)) + I(\alpha(N)) \geq I(y(N)) + I(\alpha(N)) = I(y(N) + \alpha(N)) \tag{37}$$

and

$$(I(x(N) + \alpha(N)), x[N] + \alpha[N]) \geq (I(y(N) + \alpha(N)), x[N] + \alpha[N]) \tag{38}$$

so that  $(x(N) + \alpha(N), x[N] + \alpha[N]) \succsim_U (y(N) + \alpha(N), x[N] + \alpha[N])$  by (2) and (36). Since  $\succsim_U$  is a subrelation to  $\succsim$ , we have  $(x(N) + \alpha(N), x[N] + \alpha[N]) \succsim (y(N) + \alpha(N), x[N] + \alpha[N])$ . Thus (6) is satisfied, and the Partial Unit Comparability axiom is verified.

*Sufficiency:* Suppose a SWR  $\succsim$  satisfies Axioms 1–3. We want to show that  $\succsim_U$  is a subrelation to  $\succsim$ . To this end, let  $x, y \in X$ , and suppose  $x \succ_U y$ . Then, by (1), there is some  $N' \in \mathbb{N}$ , such that  $(I(x(N')), x[N']) > (I(y(N')), y[N'])$ . So, there is  $N \geq N'$  such that

$$I(x(N)) > I(y(N)) \quad \text{and} \quad x[N] \geq y[N]$$

We want to prove that  $x \succ y$ . Denote  $[I(x(N)) - I(y(N))]$  by  $d$ ; then  $d > 0$ . Define

$$d_i = (1 - y_i)d/[N - I(y(N))] \quad \text{for } i = 1, \dots, N.$$

Note that  $0 \leq d_i \leq (1 - y_i)$  for  $i = 1, \dots, N$ , and  $\sum_{i=1}^N d_i = d$ . Now, define  $x', y', x'', y''$  as follows:  $x' = (x(N), y[N]), y' = (y_1 + d_1, \dots, y_N + d_N, y[N]), x'' = (x(N), 0[N]), y'' = (y_1 + d_1, \dots, y_N + d_N, 0[N])$ . Clearly,  $x', y' \in X$  and  $x'', y'' \in X^0$ .

Note that  $\sigma(x'') = \sigma(y'')$ , since

$$\sum_{i=1}^N (y_i + d_i) = \sum_{i=1}^N y_i + d = \sum_{i=1}^N y_i + [I(x(N)) - I(y(N))] = I(x(N)). \tag{39}$$

Using (39) and Lemma 1, we must have  $x'' \sim y''$ . Using the partial unit comparability axiom, it follows that  $x' \sim y'$ . By the Pareto Axiom, we obtain  $y' \succ y$ , and  $x \succ x'$ . Thus, we must have  $x \succ y$  by transitivity of  $\succsim$ .

Now, let  $x, y \in X$ , and suppose  $x \succsim_U y$ . Then, by (1), there is some  $N \in \mathbb{N}$ , such that

$$(I(x(N)), x[N]) \geq (I(y(N)), y[N]).$$

We want to prove that  $x \succsim y$ . If in fact we have  $(I(x(N)), x[N]) > (I(y(N)), y[N])$ , then  $x \succ_U y$ , so that  $x \succ y$  must hold, as proved above, and we are done. So, we are left with the case in which  $(I(x(N)), x[N]) = (I(y(N)), y[N])$ . In this case, define  $\bar{x}, \bar{y}$  as follows:  $\bar{x} = (x(N), 0[N]), \bar{y} = (y(N), 0[N])$ . Clearly,  $\bar{x}, \bar{y} \in X^0$ . Since  $I(x(N)) = I(y(N))$ , we have  $\sigma(\bar{x}) = \sigma(\bar{y})$

and by Lemma 1,  $\bar{x} \sim \bar{y}$ . Since  $x[N] = y[N]$ , the partial unit comparability axiom implies that  $x \sim y$ .  $\square$

**Proof of Theorem 2.** (i) Suppose a SWR  $\succsim_C$  satisfies Axioms 1–4. Let  $x, y \in X$  and  $x \succsim_C y$ . Then, by (14), there is  $\bar{N} \in \mathbb{N}$  such that

$$I(x(N)) \geq I(y(N)) \quad \text{for all } N \geq \bar{N}. \tag{40}$$

Using (40) and (2), we get  $(x(N), 0[N]) \succsim_U (y(N), 0[N])$  for all  $N \geq \bar{N}$ . Thus, for each  $N \geq \bar{N}$ , we have  $(x(N), 0[N]) \succ (y(N), 0[N])$  by using Theorem 1. Using Axiom 4(a), it follows that  $x \succ y$ .

Next, let  $x, y \in X$  and  $x \succ_C y$ . Then,  $x \succ_C y$  holds, but  $y \succ_C x$  does not hold. Using  $x \succ_C y$ , there is  $\bar{N} \in \mathbb{N}$  such that (40) holds; further (40) must hold with strict inequality for a subsequence  $N^s$  of  $N$ , otherwise  $y \succ_C x$  would also hold. Thus, using (2) and (3), we must have  $(x(N), 0[N]) \succ_U (y(N), 0[N])$  for all  $N \geq \bar{N}$ , with  $(x(N^s), 0[N^s]) \succ_U (y(N^s), 0[N^s])$  for the subsequence  $N^s$ . Consequently, we have  $(x(N), 0[N]) \succ (y(N), 0[N])$  for each  $N \geq \bar{N}$ , with  $(x(N), 0[N]) \succ (y(N), 0[N])$  for the subsequence  $N^s$ . Using Axiom 4(b), it follows that  $x \succ y$ .

(ii) Suppose that the SWR  $\succsim_C$  is a subrelation to a SWR  $\succsim$ . It can be verified, by following the method used in the proof of Theorem 1, that the SWR  $\succsim$  satisfies Axioms 1–3. We now check Axiom 4 as follows.

(a) Suppose  $x, y \in X$  and there is  $\bar{N} \in \mathbb{N}$ , such that  $(x(N), 0[N]) \succ (y(N), 0[N])$  for all  $N \geq \bar{N}$ . Pick any  $N \geq \bar{N}$ . We claim that

$$I(x(N)) \geq I(y(N)). \tag{41}$$

For if (19) is violated, then  $(y(N), 0[N]) \succ_U (x(N), 0[N])$  by (3). Using Theorem 1, we obtain  $(y(N), 0[N]) \succ (x(N), 0[N])$ , a contradiction. This establishes our claim (41). Using (41) and (15), we get  $x \succ_C y$ , and since  $\succsim_C$  is a subrelation to  $\succsim$ , we get  $x \succ y$ . This verifies Axiom 4(a).

(b) Suppose  $x, y \in X$  and there is  $\bar{N} \in \mathbb{N}$ , such that  $(x(N), 0[N]) \succ (y(N), 0[N])$  for all  $N \geq \bar{N}$ , with  $(x(N^s), 0[N^s]) \succ (y(N^s), 0[N^s])$  for a subsequence  $N^s$  of  $N \geq \bar{N}$ . Pick any  $N \geq \bar{N}$ . We can use the method used above to verify Axiom 4(a), to obtain (41), with strict inequality in (41) holding for the subsequence  $N^s$  of  $N$ . By (16),  $x \succ_C y$  holds and since  $\succsim_C$  is a subrelation to  $\succsim$ , we obtain  $x \succ y$ , verifying Axiom 4(b).  $\square$

**Proof of Theorem 4.** Our proof relies on the methods used in optimal growth theory,<sup>32</sup> and it proceeds in a number of steps,<sup>33</sup> which we now describe.

*Step 1:* Let  $\{\bar{k}_t\}$  be any maximal path from  $k_0 \in (0, \bar{k}]$ . Then, using (F.1) and (U.3), it is a standard exercise to check that  $\bar{k}_t > 0$  and  $\bar{c}_t > 0$  for  $t \geq 1$ . Further, for each  $t \geq 1$ ,  $\bar{k}_t$  must solve the following maximization problem:

$$\left. \begin{array}{l} \text{Max} \quad u[f(\bar{k}_{t-1}) - k] + u[f(k) - \bar{k}_{t+1}] \\ \text{s.t.} \quad 0 \leq k \leq f(\bar{k}_{t-1}) \\ \text{and} \quad f(k) \geq \bar{k}_{t+1}. \end{array} \right\}$$

<sup>32</sup> We rely here especially on the methods developed, for general models of intertemporal allocation, by Gale [18], McKenzie [23] and Brock [8].

<sup>33</sup> Some of the steps are very well-known in the literature, and are mentioned without further elaboration, to save space.

Since  $\bar{c}_t > 0$  and  $\bar{c}_{t+1} > 0$ , the solution is an interior one, and so the following first-order condition must hold

$$u'(\bar{c}_t) = u'(\bar{c}_{t+1})f'(\bar{k}_t) \quad \text{for } t \geq 1. \tag{42}$$

Thus,  $\{\bar{k}_t\}$  is a *Ramsey–Euler path*.

Define the price sequence  $\{\bar{p}_t\}$  by

$$\bar{p}_t = u'(\bar{c}_t) \quad \text{for } t \geq 1, \quad \bar{p}_0 = u'(\bar{c}_1)f'(k_o). \tag{43}$$

Then, using the concavity and differentiability of  $f$  and  $u$ , it is easy to check that for  $t \geq 0$

$$\left. \begin{aligned} \bar{p}_{t+1}f(\bar{k}_t) - \bar{p}_t\bar{k}_t &\geq \bar{p}_{t+1}f(k) - \bar{p}_tk & \text{for all } k \geq 0, \\ u(\bar{c}_t) - \bar{p}_t\bar{c}_t &\geq u(c) - \bar{p}_tc & \text{for all } c \geq 0. \end{aligned} \right\} \tag{44}$$

That is,  $\{\bar{k}_t\}$  is a *competitive path*. Since  $\{\bar{k}_t\}$  is a maximal path, and  $u$  is increasing, there is no path  $\{k'_t\}$  from  $k_o$  satisfying

$$c'_t \geq \bar{c}_t \quad \text{for all } t \geq 1, \quad \text{and } c'_t > \bar{c}_t \quad \text{for some } t \geq 1.$$

Thus,  $\{\bar{k}_t\}$  is an *efficient path*, and by the result of Brock [10], we have

$$\liminf_{T \rightarrow \infty} \sum_{t=1}^T [u(c'_t) - u(\bar{c}_t)] \leq 0 \tag{45}$$

for every feasible path  $\{k'_t\}$  from  $k_o$ .

*Step 2:* Given the golden-rule stock  $\hat{k}$ , define  $\hat{c} = f(\hat{k}) - \hat{k}$ ; it is easy to check that  $\hat{c} > 0$ . We now come to a key concept in our demonstration, that of a *good path*. A feasible path  $\{k_t\}$  is called *good* if there is  $G \in \mathbb{R}$  and  $N \in \mathbb{N}$  such that

$$\sum_{t=1}^T [u(c_t) - u(\hat{c})] \geq G \quad \text{for all } T > N. \tag{46}$$

It can be verified that there is a feasible path  $\{\tilde{k}_t\}$  from  $k_o$  and  $N \in \mathbb{N}$  such that  $\tilde{k}_t = \hat{k}$  for all  $t \geq N$ , and so  $\tilde{c}_{t+1} = \hat{c}$  for all  $t \geq N$ . This is clearly a good path from  $k_o$ .

*Step 3:* Next, define the golden-rule price  $\hat{p} = u'(\hat{c})$ . Then, using the concavity and differentiability of  $f$  and  $u$ , it can be verified that

$$\left. \begin{aligned} \hat{p}f(\hat{k}) - \hat{p}\hat{k} &\geq \hat{p}f(k) - \hat{p}k & \text{for all } k \geq 0, \\ u(\hat{c}) - \hat{p}\hat{c} &\geq u(c) - \hat{p}c & \text{for all } c \geq 0. \end{aligned} \right\} \tag{47}$$

Define  $\Omega = \{(k, k') \in \mathbb{R}_+^2 : k' \leq f(k)\}$  and a felicity function  $w : \Omega \rightarrow \mathbb{R}$  by  $w(k, k') = u(f(k) - k')$ . Then the condition (47) can be combined to yield for  $t \geq 0$

$$w(\hat{k}, \hat{k}) \geq w(k, k') + \hat{p}k' - \hat{p}k \quad \text{for all } (k, k') \in \Omega. \tag{48}$$

For  $(k, k') \in \Omega$ , denote  $[w(\hat{k}, \hat{k}) - \{w(k, k') + \hat{p}k' - \hat{p}k\}]$  by  $\delta(k, k')$ . Then  $\delta(k, k') \geq 0$  by (48).

Step 4: We now use (48) to establish an important characterization of good paths. Let  $\{k_t\}$  be any feasible path from  $k_o$ . Then, noting that  $(k_t, k_{t+1}) \in \Omega$  for  $t \geq 0$ , and denoting  $\delta(k_t, k_{t+1})$  by  $\delta_t$  for  $t \geq 0$ , we have

$$\{k_t\} \text{ is a good path if } f \sum_{t=0}^{\infty} \delta_t < \infty. \tag{49}$$

To see this, note that for  $T \in \mathbb{N}$

$$\begin{aligned} \sum_{t=0}^T [w(k_t, k_{t+1}) - w(\hat{k}, \hat{k})] &= \sum_{t=0}^T [\hat{p}k_t - \hat{p}k_{t+1}] - \sum_{t=0}^T \delta_t \\ &= \hat{p}k_o - \hat{p}k_{T+1} - \sum_{t=0}^T \delta_t. \end{aligned} \tag{50}$$

If  $\sum_{t=0}^{\infty} \delta_t < \infty$ , then using (34), the right-hand side expression is bounded below by a real number, independent of  $T$ , and so  $\{k_t\}$  is good. Conversely, if  $\{k_t\}$  is good, then there is  $G \in \mathbb{R}$  such that the left-hand side of (50) is bounded below by  $G$  independent of  $T$ , and so  $\sum_{t=0}^T \delta_t$  is bounded above by  $[\hat{p}k_o - G]$ , independent of  $T$ ; thus,  $\sum_{t=0}^{\infty} \delta_t < \infty$  must hold. A consequence of this characterization result is that if  $\{k_t\}$  is a feasible path from  $k_o$ , which is *not* good, then (using (50)):

$$\sum_{t=0}^T [w(k_t, k_{t+1}) - w(\hat{k}, \hat{k})] \rightarrow -\infty \text{ as } T \rightarrow \infty. \tag{51}$$

Step 5: Part of the above characterization result leads to the basic asymptotic properties of good paths. If  $\{k_t\}$  is any good path from  $k_o$ , then, noting that  $(k_t, k_{t+1}) \in \Omega$  for  $t \geq 0$ , and denoting  $\delta(k_t, k_{t+1})$  by  $\delta_t$  for  $t \geq 0$ , we have

$$\left. \begin{aligned} \text{(i)} \quad &\delta_t \rightarrow 0 \text{ as } t \rightarrow \infty, \\ \text{(ii)} \quad &(k_t, c_t) \rightarrow (\hat{k}, \hat{c}) \text{ as } t \rightarrow \infty. \end{aligned} \right\} \tag{52}$$

While (52)(i) follows directly from (49), (52)(ii) follows from (52)(i) and the “value-loss lemma”, originally due to Radner [27], which we now proceed to discuss in the context of our framework. Note that  $w(k, k')$ , and therefore,  $\delta(k, k')$  is continuous on  $\Omega$ , and further that  $\delta(k, k') > 0$  whenever  $(k, k') \neq (\hat{k}, \hat{k})$ , by strict concavity of  $f$  and  $u$ . For  $0 < \varepsilon < \hat{k}$ , define the set

$$S(\varepsilon) = \{(k, k') \in \Omega : k \leq \bar{k}, k' \leq \bar{k}, \text{ and } |k - \hat{k}| + |k' - \hat{k}| \geq \varepsilon\}.$$

Now,  $S(\varepsilon)$  is a non-empty, compact set in  $\mathbb{R}^2$ , and  $\delta(k, k')$  is continuous on  $S(\varepsilon)$ . So there is some  $(\mathbf{k}, \mathbf{k}') \in S(\varepsilon)$ , such that  $\delta(k, k') \geq \delta(\mathbf{k}, \mathbf{k}')$  for all  $(k, k') \in S(\varepsilon)$ . Denoting  $\delta(\mathbf{k}, \mathbf{k}')$  by  $\delta$ , we see that  $\delta > 0$ , since  $(\mathbf{k}, \mathbf{k}') \neq (\hat{k}, \hat{k})$ . Thus,  $\delta(k, k') \geq \delta > 0$  for all  $(k, k') \in S(\varepsilon)$ .

Applying the above “value-loss” result to  $(k_t, k_{t+1}) \in \Omega$ , we must have  $(k_t, k_{t+1}) \rightarrow (\hat{k}, \hat{k})$  as  $t \rightarrow \infty$ , by (52)(i). It then follows that  $c_t = f(k_{t-1}) - k_t \rightarrow f(\hat{k}) - \hat{k} = \hat{c}$  as  $t \rightarrow \infty$ . This establishes (52)(ii).

Step 6: We now observe, using (45), that  $\{\bar{k}_t\}$  must be a good path from  $k_o$ . For, if  $\{\bar{k}_t\}$  is not a good path from  $k_o$ , then noting that  $\{\tilde{k}_t\}$  is a good path from  $k_o$ , there is  $G \in \mathbb{R}$ , and  $N \in \mathbb{N}$ ,

such that, for  $T > N$

$$\begin{aligned} \sum_{t=0}^T [w(\tilde{k}_t, \tilde{k}_{t+1}) - w(\bar{k}_t, \bar{k}_{t+1})] &= \sum_{t=0}^T [w(\tilde{k}_t, \tilde{k}_{t+1}) - w(\hat{k}, \hat{k})] \\ &\quad + \sum_{t=0}^T [w(\hat{k}, \hat{k}) - w(\bar{k}_t, \bar{k}_{t+1})] \\ &\geq G + \sum_{t=0}^T [w(\hat{k}, \hat{k}) - w(\bar{k}_t, \bar{k}_{t+1})]. \end{aligned} \tag{53}$$

The expression in the third line of (53) goes to  $\infty$  as  $T \rightarrow \infty$  by using (51). Thus, the left-hand side expression in the first line of (53) goes to  $\infty$  as  $T \rightarrow \infty$ . But this would contradict the property (45) already established for  $\{\bar{k}_t\}$ .

Step 7: The conditions (44) can be combined to yield for  $t \geq 0$ :

$$w(\bar{k}_t, \bar{k}_{t+1}) + \bar{p}_{t+1}\bar{k}_{t+1} - \bar{p}_t\bar{k}_t \geq w(k, k') + \bar{p}_{t+1}k' - \bar{p}_tk \quad \text{for all } (k, k') \in \Omega \tag{54}$$

a generalized profit-maximization property, at the prices  $\{\bar{p}_t\}$ . For  $(k, k') \in \Omega$ , denote  $[w(\bar{k}_t, \bar{k}_{t+1}) + \bar{p}_{t+1}\bar{k}_{t+1} - \bar{p}_t\bar{k}_t - \{w(k, k') + \bar{p}_{t+1}k' - \bar{p}_tk\}]$  by  $L(k, k')$ . Then  $L(k, k') \geq 0$  by (54). In this final step, we use (54) to establish that  $\{\bar{k}_t\}$  overtakes every other feasible path from  $k_o$ . Let  $\{k'_t\}$  be any feasible path from  $k_o$ , which is distinct from  $\{\bar{k}_t\}$ . Noting that  $(k'_t, k'_{t+1}) \in \Omega$  for  $t \geq 0$ , denote  $L(k_t, k_{t+1})$  by  $L'_t$  for  $t \geq 0$ . Let  $\tau$  be the first period when the paths differ; that is,  $k'_\tau \neq \bar{k}_\tau$ , but  $k'_t = \bar{k}_t$  for  $t = 0, \dots, \tau - 1$ . Then,  $c'_\tau \neq \bar{c}_\tau$ , and so by using the strict concavity of  $u$ , we would get  $L'_\tau > 0$ . We now consider two cases (i)  $\{k'_t\}$  is not good; (ii)  $\{k'_t\}$  is good. In case (i), we write for  $T \in \mathbb{N}$

$$\begin{aligned} \sum_{t=0}^T [w(k'_t, k'_{t+1}) - w(\bar{k}_t, \bar{k}_{t+1})] &= \sum_{t=0}^T [w(k'_t, k'_{t+1}) - w(\hat{k}, \hat{k})] \\ &\quad + \sum_{t=0}^T [w(\hat{k}, \hat{k}) - w(\bar{k}_t, \bar{k}_{t+1})] \end{aligned} \tag{55}$$

and note that since  $\{\bar{k}_t\}$  is good, the second expression on the right-hand side of (55) is bounded above, independent of  $T$ , while since  $\{k'_t\}$  is not good, the first expression on the right-hand side of (55) goes to  $-\infty$  as  $T \rightarrow \infty$ , by (51). Thus, the left-hand side expression of (55) must go  $-\infty$  as  $T \rightarrow \infty$ . Consequently,  $\{\bar{k}_t\}$  overtakes  $\{k'_t\}$ .

In case (ii), we have for  $T \in \mathbb{N}$ , with  $T > \tau$ :

$$\begin{aligned} \sum_{t=0}^T [w(k'_t, k'_{t+1}) - w(\bar{k}_t, \bar{k}_{t+1})] &= \sum_{t=0}^T [\bar{p}_tk'_t - \bar{p}_{t+1}k'_{t+1}] \\ &\quad - \sum_{t=0}^T [\bar{p}_t\bar{k}_t - \bar{p}_{t+1}\bar{k}_{t+1}] - \sum_{t=0}^T L_t \\ &= [\bar{p}_{t+1}\bar{k}_{t+1} - \bar{p}_{t+1}k'_{t+1}] - \sum_{t=0}^T L_t \\ &\leq [\bar{p}_{t+1}\bar{k}_{t+1} - \bar{p}_{t+1}k'_{t+1}] - L_\tau. \end{aligned} \tag{56}$$

Now, since  $\{\bar{k}_t\}$  and  $\{k'_t\}$  are both good, we have  $k'_t \rightarrow \hat{k}$ ,  $\bar{k}_t \rightarrow \hat{k}$  as  $t \rightarrow \infty$ , and further  $\bar{c}_t \rightarrow \hat{c}$  as  $t \rightarrow \infty$ , by (52), so that  $\bar{p}(t) = u'(\bar{c}_t) \rightarrow u'(\hat{c}) = \hat{p}$  as  $t \rightarrow \infty$ . Thus, there is  $N > \tau$ , such that for all  $T > N$ , we have

$$[\bar{p}_{t+1}\bar{k}_{t+1} - \bar{p}_{t+1}k'_{t+1}] < (L_\tau/2). \quad (57)$$

Then, using (56) and (57), we have for all  $T > N$ ,

$$\sum_{i=0}^T [w(k'_i, k'_{i+1}) - w(\bar{k}_i, \bar{k}_{i+1})] < -(L_\tau/2)$$

so that  $\{\bar{k}_t\}$  overtakes  $\{k'_t\}$ .  $\square$

## References

- [1] G.B. Asheim, W. Buchholz, B. Tungodden, Justifying sustainability, *J. Environ. Econ. Manage.* 41 (2001) 252–268.
- [2] G.B. Asheim, B. Tungodden, Resolving distributional conflicts between generations, *Econ. Theory* 24 (2004) 221–230.
- [3] H. Atsumi, Neoclassical growth and the efficient program of capital accumulation, *Rev. Econ. Stud.* 32 (1965) 127–136.
- [4] K. Banerjee, On the extension of the utilitarian and Suppes-Sen social welfare relations to infinite utility streams, *Social Choice Welfare*, 2005, forthcoming.
- [5] K. Basu, Cardinal utility, utilitarianism and a class of invariance axioms in welfare analysis, *J. Math. Econ.* 12 (1983) 193–206.
- [6] K. Basu, T. Mitra, Aggregating infinite utility streams with intergenerational equity: the impossibility of being paretian, *Econometrica* 71 (2003) 1557–1563.
- [7] C. Blackorby, D. Donaldson, J.A. Weymark, Social choice with interpersonal utility comparisons: a diagrammatic introduction, *Int. Econ. Rev.* 25 (1984) 327–356.
- [8] W.A. Brock, On the existence of weakly maximal programmes in a multi-sector economy, *Rev. Econ. Stud.* 37 (1970a) 275–280.
- [9] W.A. Brock, An axiomatic basis for the Ramsey–Weizsacker overtaking criterion, *Econometrica* 38 (1970b) 927–929.
- [10] W.A. Brock, Sensitivity of optimal growth paths with respect to a change in target stocks, in: G. Bruckmann, W. Weber (Eds.), *Contributions to the von Neumann Growth Model*, Springer, Vienna, 1971, pp. 73–90.
- [11] C. d'Aspremont, Axioms for social welfare orderings, in: L. Hurwicz, D. Schmeidler, H. Sonnenschein (Eds.), *Social Goals and Social Organizations: Essays in Memory of Elisha Pazner*, Cambridge University Press, Cambridge, 1985, pp. 19–76.
- [12] C. d'Aspremont, L. Gevers, Equity and the informational basis of collective choice, *Rev. Econ. Stud.* 46 (1977) 199–210.
- [13] G. Debreu, *Theory of Value*, Wiley, New York, 1959.
- [14] G. Debreu, Topological methods in cardinal utility theory, in: K.J. Arrow, S. Karlin, P. Suppes (Eds.), *Mathematical Methods in the Social Sciences*, Stanford University Press, Stanford, 1960.
- [15] P. Diamond, The evaluation of infinite utility streams, *Econometrica* 33 (1965) 170–177.
- [16] P.C. Fishburn, *Utility Theory for Decision Making*, Wiley, New York, 1970.
- [17] M. Fleurbaey, P. Michel, Intertemporal equity and the extension of the Ramsey principle, *J. Math. Econ.* 39 (2003) 777–802.
- [18] D. Gale, On optimal development in a multi-sector economy, *Rev. Econ. Stud.* 34 (1967) 1–18.
- [19] T.C. Koopmans, Stationary ordinal utility and impatience, *Econometrica* 28 (1960) 287–309.
- [20] T.C. Koopmans, P.A. Diamond, R.E. Williamson, Stationary utility and time perspective, *Econometrica* 32 (1964) 82–100.
- [21] L. Lauwers, Rawlsian equity and generalized utilitarianism with an infinite population, *Econ. Theory* 9 (1997) 143–150.
- [22] E. Maskin, A theorem on utilitarianism, *Rev. Econ. Stud.* 45 (1978) 93–96.
- [23] L.W. McKenzie, Accumulation programs of maximum utility and the von Neumann facet, in: J.N. Wolfe (Ed.), *Value, Capital and Growth*, Edinburgh University Press, Edinburgh, 1968, pp. 353–383.

- [24] J. Milnor, *Games Against Nature*, Chapter IV, in: R.M. Thrall, C.H. Coombs, R.L. Davis (Eds.), *Decision Processes*, Wiley, New York, 1954(Chapter IV).
- [25] T. Mitra, Intergenerational equity and the forest management problem, in: S. Kant, R.A. Berry (Eds.), *Economics, Sustainability, and Natural Resources: Economics of Sustainable Forest Management*, Springer, Dordrecht, 2005, pp. 137–173.
- [26] T. Mitra, K. Basu, On the existence of paretian social welfare quasi orderings for infinite utility streams with extended anonymity, in: J. Roemer, K. Suzumura (Eds.), *Intergenerational Equity and Sustainability*, Palgrave Macmillan, Oxford, 2005, forthcoming.
- [27] R. Radner, Paths of economic growth that are optimal only with respect to final states, *Rev. Econ. Stud.* 28 (1961) 98–104.
- [28] F.P. Ramsey, A mathematical theory of savings, *Economic Journal* 38 (1928) 543–559.
- [29] K.W.S. Roberts, Interpersonal comparability and social choice theory, *Rev. Econ. Stud.* 47 (1980) 421–439.
- [30] A.K. Sen, *Collective Choice and Social Welfare*, Oliver & Boyd, Edinburgh, 1971.
- [31] A.K. Sen, On weights and measures: informational constraints in social welfare analysis, *Econometrica* 45 (1977) 1539–1572.
- [32] P. Suppes, Some formal models of grading principles, *Synthese* 6 (1966) 284–306.
- [33] L.-G. Svensson, Equity among generations, *Econometrica* 48 (1980) 1251–1256.
- [34] C.C. von Weizsäcker, Existence of optimal programs of accumulation for an infinite time horizon, *Rev. Econ. Stud.* 32 (1965) 85–104.